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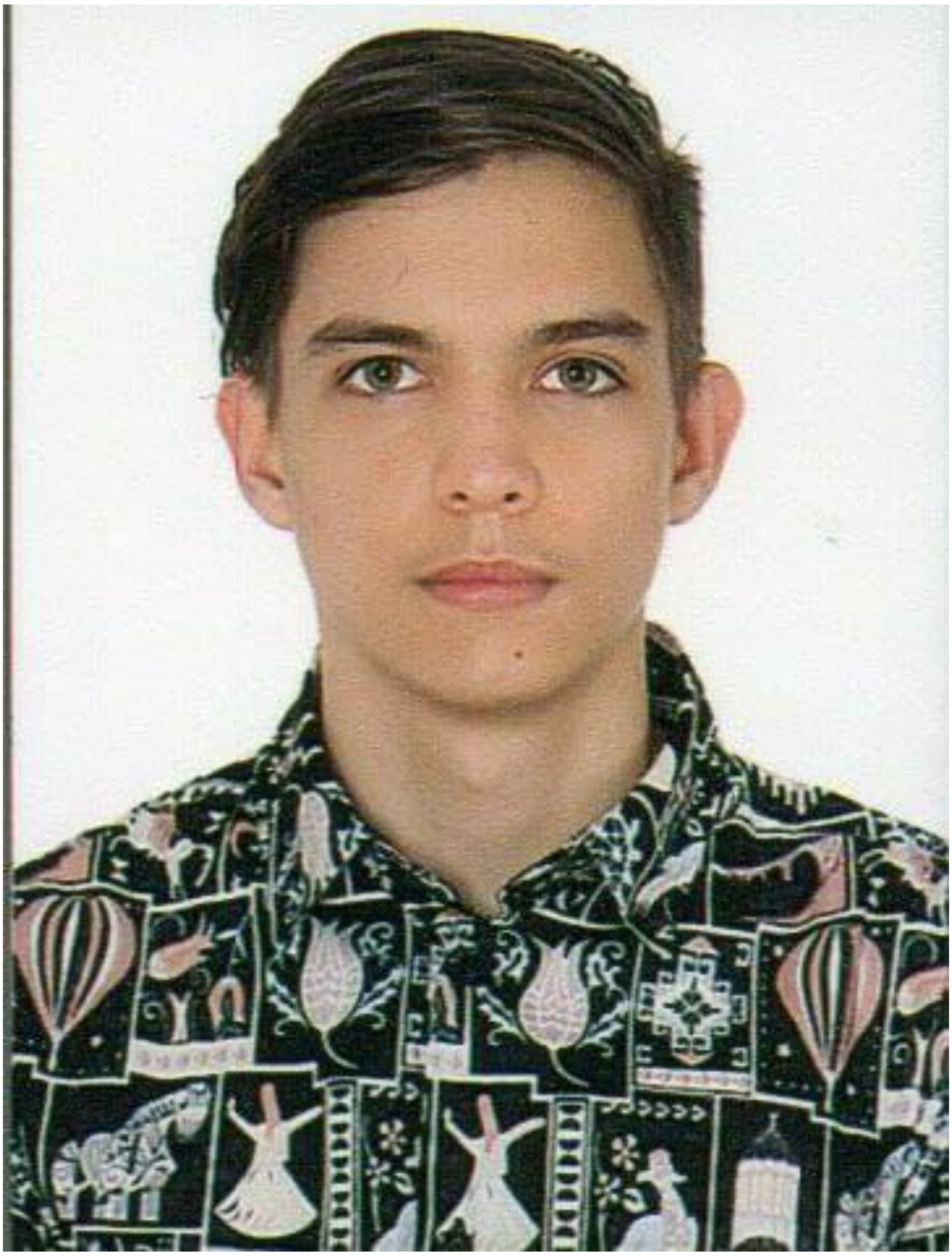

ABSTRACT

The project is devoted to the study of the Seymour’s Second Neighborhood conjecture by determining the properties of possible counterexamples to it. This problem has remained unsolved for more than 30 years, although there is some progress in its solution.

The vector of the research is aimed at the analysis of possible counterexamples to the conjecture with the subsequent finding of some of their characteristic values. In addition, attention is focused on the generalized Seymour’s conjecture for vertex-weighted graphs. Combinatorial research methods and graph theory methods were used in the project.

The author determines the values of densities and diameters of possible counterexamples, considers separately directed graphs of diameter 3. The conditions under which specific graphs cannot be counterexamples to the Seymour’s conjecture with the minimum number or vertices are defined. The relationship between the Seymour’s conjecture and vertex-weighted Seymour’s conjecture is explained.

It is proved that if there exists at least one counterexample, then there exist counterexamples with an arbitrary diameter not less than 3. Under the same condition, the existence of counterexamples with a density both close to 0 and close to 1 is also proved. The equivalence of the above two conjectures is substantiated in detail.

It can be concluded that if the Seymour’s Second Neighborhood Conjecture is true for a directed graph of diameter 3, then it is true for any digraph, so that problem will be solved. Moreover, if the conjecture is true, then vertex-weighted version of this conjecture is true too. That is why a digraph of diameter 3 needs further research.

Keywords: directed graph, Seymour's Second Neighborhood Conjecture, density, diameter, vertex-weighted graph.

1. INTRODUCTION

Seymour's Second Neighborhood Conjecture is one of the most famous unsolved mathematical problems in the theory of graphs, which is formulated by Paul Seymour in 1990.

The relevance of the chosen project topic is determined by the rapid development of graph theory, which is associated with expansion of its scope of usage: business, logistics, tourism and most importantly the creation of computer programs.

The goal of the project is to investigate properties of possible counterexamples to the Seymour's conjecture.

According to the goal, the objectives of the project are as follows:

- to analyse the current state of research of the Seymour's conjecture;
- to discuss the range of values of diameters and densities that characterize possible counterexamples as graphs;
- to prove that graphs with certain properties cannot be counterexamples to the Seymour's conjecture with the minimum number or vertices, using the penalty function;
- to establish a connection between Seymour's conjecture and its vertex-weighted version.

The object of project is a mathematical problem of graph theory – Seymour's conjecture.

The subject of the project is possible counterexamples to the Seymour's conjecture.

Combinatorial research methods and graph theory methods were used during the research.

All digraphs considered in this project are oriented simple graphs (orgraphs), thus do not contain cycles of length two or loops, i.e., a graph cannot contain two arcs (v, u) and (u, v) at the same time.

Seymour's Second Neighborhood Conjecture. In any oriented graph D without cycles of length two there exists a vertex $v \in V(D)$ for which $d^{++}(v) \geq d^{+}(v)$.

Such a vertex v is called the Seymour vertex.

It was proved that this conjecture holds if D is a tournament graph [5]. In addition, a generalized version of Seymour's conjecture for arc-weighted oriented graphs was considered in [6].

The maximum value of some number γ was also determined, for which in every oriented graph D there exists a vertex $v \in V(D)$, for which $d^{++}(v) \geq \gamma d^{+}(v)$. This inequality was proved for $\gamma = 0.657298 \ldots$, where γ is the only real root of the equation $2x^3 + x^2 - 1 = 0$ [7].

The results of section 3 were submitted by me and my colleagues to a scientific journal and were published recently [10].

2. DEFINITIONS

Definition 2.1. A graph $G = (V, E)$ is a pair of sets (V, E), where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. The set of vertices of the graph G is usually denoted by $V(G)$ and the set of edges by $E(G)$ [2, p. 61].

Definition 2.2. A graph is called simple if no more than one edge connects each pair of vertices. A graph is called a multigraph if it has multiple edges. A graph is called a pseudograph if it has loops and multiple edges [1, p. 2].

Definition 2.3. The number of edges that incident on the vertex v is called a degree (or valence) of the vertex v and is denoted by $d(v)$. If the degree of the vertex is zero ($d(v) = 0$) then the vertex is isolated [2, p. 62].

Definition 2.4. The sum of the degrees of the vertices of a graph is twice the number of its edges [2, p. 62].

Definition 2.5. A directed graph (digraph) is a graph $D = (V, A)$ where V is the set of vertices and, $A \subseteq V \times V$ is the set of arcs [1, p. 70]. The direction of the arcs is indicated by arrows in graphs.

Definition 2.6. For a vertex v in a digraph the number of arcs that go from a vertex is called outdegree of v, and the number of arcs that go in a vertex is called indegree of v. Outdegree and indegree are denoted by $d^{+}(v)$ and $d^{-}(v)$ respectively [3, p. 47].

Definition 2.7. A walk in an undirected graph G is a sequence of alternating vertices and edges: $(v_1, e_1, v_2, e_2, ..., v_{n-1}, e_{n-1}, v_n)$, where every two adjacent edges e_{i-1} and e_i have a common incident vertex v_i [3, p. 52].

Definition 2.8. A walk is called a trail if each edge occurs in it no more than once, and called a path if any vertex (except, perhaps, initial) meets in it no more than once. A trail is called a cycle if it is closed, and a path is called a simple cycle if it is closed. A graph that does not contain cycles is called an acyclic [3, p. 53].

In a directed graph a walk, a trail and a path are called a directed walk, a directed trail and a directed path respectively.

Definition 2.9. A digraph is strongly connected if for any two of its vertices v and w there is a directed path in both directions [3, p. 55].

Definition 2.10. A graph $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$ [2, p. 63]. A similar definition of a subgraph is used for digraphs.

Definition 2.11. The distance $d(v, w)$ in a graph between two vertices v and w is the length of shortest path connecting the vertices v i w [2, p. 65]. If there is no such path, we set $d(v, w) = \infty$. If the distance is considered only in a certain subgraph S of the graph D then this subgraph is indicated in brackets after a pair or vertices, for example $d(i, j, S)$.

Definition 2.12. The diameter of connected graph is the maximum of all eccentricities of the vertices of the graph. The eccentricity of an arbitrary vertex v of connected graph is the largest of the distances between the vertex v and all other vertices of the graph [2, p. 65]. In this project, the diameter of the graph D is denoted by $diam(D)$.

Definition 2.13. $N^-(v)$, $N^+(v)$, $N^{++}(v)$ and $N^{+++}(v)$ are the sets of all vertices w for which $d(w, v) = 1$, $d(v, w) = 1$, $d(v, w) = 2$ and $d(v, w) = 3$ respectively.

$d^-(v) = |N^-(v)|$, $d^+(v) = |N^+(v)|$, $d^{++}(v) = |N^{++}(v)|$, $d^{+++}(v) = |N^{+++}(v)|$. If this is considered only in a certain subgraph S of the graph D then this subgraph is indicated in brackets after the vertex, for example $d^+(\mathrm{v}, S)$.

Definition 2.14. The density ρ of a graph is the ratio of the number of its edges (arcs) to the number of all possible edges (arcs) of the graph [4, p. 148].

Definition 2.15. The penalty function H is defined on the vertex v and on the digraph D as follows:

$$H(v) = \begin{cases} d^{++}(v) - d^{+}(v) + 1, \text{if } d^{++}(v) \geq d^{+}(v) \\ 0, \text{if } d^{++}(v) < d^{+}(v) \end{cases}$$

$$H(D) = \sum_{v \in V(D)} H(v)$$

$H(v)$ is called the penalty of the vertex v and $H(D)$ is called the penalty of digraph D. If $H(D) > 0$ then D has a Seymour vertex.

3. THE PROBLEM OF DENSITY AND DIAMETER OF COUNTEREXAMPLES

Since we consider digraphs without cycles of length two, the maximum number of arcs in such graphs is equal to $\frac{n(n-1)}{2}$. Therefore $\rho = \frac{2|A|}{n(n-1)}$, $0 \le \rho \le 1$.

For arbitrary $k \ge 3$ ($k \in \mathbb{N}$) and an orgraph D with n vertices, let $D^{(1)}$, $D^{(2)}$, …, $D^{(k)}$ be vertex disjoint copies of D. Let $D(k)$ be the orgraph with the set of vertices $V\big(D(k)\big) = \bigcup_{i=1}^{k} V(D^{(i)})$ and the arc set equal to

$$A\big(D(k)\big) = \bigcup_{i=1}^{k} A(D^{(i)}) \cup \{(u,v): u \in V\big(D^{(i)}\big), v \in V\big(D^{(i+1)}\big), i = 1, 2, \ldots, k-1\} \cup$$
$$\cup \{(u,v): u \in V\big(D^{(k)}\big), v \in V\big(D^{(1)}\big)\}$$

We will use i-th level of $D(k)$ to refer to $D^{(i)}$ of $D(k)$. It can be noticed that orgraph $D(k)$ is strongly connected.

From the given definition it follows that $\left|V\big(D(k)\big)\right| = kn$, because the graph contains k subgraphs, and $d^{+}\big(v, D(k)\big) = d^{+}(v, D) + n$ та $d^{++}\big(v, D(k)\big) = d^{++}(v, D) + n$ for an arbitrary vertex $v \in V(D)$ (regardless of the level at which this vertex v is located). Therefore, if D has a Seymour vertex v, i.e., $d^{++}(v, D) \ge d^{+}(v, D)$, then $D(k)$ also has a Seymour vertex, since $d^{++}(v, D) + n \ge d^{+}(v, D) + n \Longrightarrow d^{++}\big(v, D(k)\big) \ge d^{+}\big(v, D(k)\big)$. $D(k)$ will have at least k Seymour vertices, because each level of $D(k)$ has at least one Seymour vertex.

Lemma 3.1. If D is a counterexample to the Seymour's conjecture then also $D(k)$ is a counterexample to the Seymour's conjecture.

Proof. Let $D = (V; A)$ be a counterexample to the conjecture with n vertices. Consider arbitrary $v \in V\big(D(k)\big)$. Without loss of generality let v be on the first level of $D(k)$. First level of $D(k)$ is a counterexample to the conjecture, so $d^{+}(v, D) > d^{++}(v, D)$. Since $d^{+}\big(v, D(k)\big) = d^{+}(v, D) + n$ and $d^{++}\big(v, D(k)\big) = d^{++}(v, D) + n$, we obtain $d^{+}\big(v, D(k)\big) > d^{++}(v, D(k))$. Vertex v was selected arbitrarily, thus $D(k)$ is a counterexample to the Seymour's conjecture.

Theorem 3.2. If there exists a counterexample D to the Seymour's conjecture, then for any $\varepsilon > 0$ there exists a strongly connected counterexample D^* with a density less than ε.

Proof. Let $D = (V; A)$ be a counterexample to the conjecture with n vertices. From Lemma 3.1 it follows that $D^* = D(k)$ is a counterexample too. The number or vertices in this graph is equal to $|V(D^*)| = kn$. At each level no more than $\frac{n(n-1)}{2}$ arcs, and exactly n^2 arcs go to the next level, then the total number or arcs of the graph D^* does not exceed $k\left(\frac{n(n-1)}{2} + n^2\right)$, i.e., $|A(D^*)| \leq k\left(\frac{n(n-1)}{2} + n^2\right)$.

Using the previous inequality, we obtain an estimation for the density ρ:

$$\rho(D^*) \leq \frac{2\left(\frac{n(n-1)}{2} + n^2\right)k}{kn(kn-1)} = \frac{3n-1}{kn-1}$$

Inequality $\frac{3n-1}{kn-1} < \varepsilon$ is equivalent to $k > \frac{3n+\varepsilon-1}{\varepsilon n}$. Since $k \geq 3$, put $k = \left\lceil \frac{3n+\varepsilon-1}{\varepsilon n} \right\rceil + 2$. We obtain the inequality $\rho(D^*) \leq \frac{3n-1}{kn-1} < \varepsilon$, which had to be proved. Theorem 3.2 is proved.

Theorem 3.3. If there exists a counterexample D to the Seymour's conjecture, then for any $\varepsilon > 0$ there exists a strongly connected counterexample D^{**} with a density greater than $1 - \varepsilon$.

Proof. Consider a counterexample $D = (V; A)$ which has n vertices. From Lemma 3.1 the digraph $D(3)$ is a counterexample to the Seymour's conjecture, which has $3n$ vertices and at least $3n^2$ arcs, because between each pair of levels there are exactly n^2 arcs. That is, $\left|A\big(D(3)\big)\right| \geq 3n^2$, therefore

$$\rho\big(D(3)\big) \geq \frac{2 \cdot 3n^2}{3n(3n-1)} > \frac{2 \cdot 3n^2}{3n \cdot 3n} = \frac{2}{3} \tag{3.1}$$

The sequence of orgraphs (H_i) is defined recursively:

$H_1 = D(3)$,

$H_{k+1} = H_k(3)$ for $k \geq 1$.

All the orgraphs H_k are strongly connected counterexamples to the Seymour's conjecture (from Lemma 3.1), for which $|V(H_k)| = 3^k n$, because each subsequent level has 3 times more vertices.

Lemma 3.4. For every $k \in \mathbb{N}$: $\rho(H_k) \geq 1 - \frac{1}{3^k}$.

Proof. For $k = 1$, Lemma 3.4 follows by inequality (3.1). For $k \geq 2$ using [3, c. 47] we will have the following

$$|A(H_k)| = \frac{1}{2}\left(\sum_{v \in H_k} d^+(v) + \sum_{v \in H_k} d^-(v)\right) = \frac{1}{2}\sum_{v \in H_k}(d^+(v) + d^-(v)) \tag{3.2}$$

By construction of the orgraph H_k

$$d^+(v) + d^-(v) \geq 3^k n - n \tag{3.3}$$

since vertex v can be not adjacent only to vertices of subgraph D.

Substitute (3.3) into (3.2) to obtain

$$|A(H_k)| = \frac{1}{2}\sum_{v \in H_k}(d^+(v) + d^-(v)) \geq \frac{1}{2}\sum_{v \in H_k}(3^k n - n) = \frac{1}{2}3^k n(3^k n - n)$$

and finally

$$\rho(H_k) = \frac{2|A(H_k)|}{3^k n(3^k n - 1)} \geq \frac{2|A(H_k)|}{3^k n(3^k n)} \geq \frac{2\left(\frac{1}{2}3^k n(3^k n - n)\right)}{3^k n(3^k n)} = \frac{3^k n - n}{3^k n} = 1 - \frac{1}{3^k}$$

Lemma 3.4 is proved.

Consider $D^{**} = H_k$. Let's solve the inequality $1 - \frac{1}{3^k} > 1 - \varepsilon$ for k:

$$1 - \frac{1}{3^k} > 1 - \varepsilon \iff 3^k > \frac{1}{\varepsilon} \iff k > \log_3 \frac{1}{\varepsilon}$$

$\rho(D^{**}) > 1 - \varepsilon$ holds when $k > \log_3 \frac{1}{\varepsilon}$, which completes the proof of the Theorem 3.3.

Theorem 3.5. If there exists a counterexample D to the Seymour's conjecture, then for any $k \geq 3$ *(*$k \in \mathbb{N}$) there exists a counterexample D^{***}, for which $diam(D^{***}) = k$.

Proof. Consider the digraph $D = (V; A)$, which is a counterexample to the conjecture. There are 2 possible cases.

Case 1. $diam(D) \geq k$. Consider digraph $D^{***} = D(k)$, which by Lemma 3.1 will be a counterexample to the conjecture. Consider an arbitrary pair of vertices $i, j \in V(D^{***})$.

$$d(i, j, D^{***}) \leq k, \tag{3.4}$$

since in 1 step we can go from the vertex i to the vertex of the next level, in k steps we can go through all levels and return to the level with the vertex i, i.e., in k steps we can reach any vertex of the graph D^{***}. Since $diam(D) \geq k,$ then for some $i, j \in V\left(D^{(q)}\right),\ 1 \leq q \leq k,$ the following inequality holds $d\left(i, j, D^{(q)}\right) \geq k.$

From the previous statement we obtain

$$d(i, j, D^{***}) \geq d\left(i, j, D^{(q)}\right) \geq k \tag{3.5}$$

From inequalities (3.4) and (3.5) it follows that $diam(D^{***}) = k.$

Case 2. $diam(D) < k$. Consider digraph $D^{***} = D(k+1)$, which by Lemma 3.1 will be a counterexample to the conjecture. From each vertex you can go through all the levels in k steps, but you can't go back to the starting level. Therefore, for vertices at different levels it is true that $d(i, j, D^{***}) \leq k$. If the vertices are at the same level, then from $diam(D) < k$ it follows that $d\left(i, j, D^{(q)}\right) < k,\ 1 \leq q \leq k.$

So, for an arbitrary pair of vertices $i, j \in V(D^{***})$ it is true that $d(i, j, D^{*}) \leq k$. If i is the vertex of the first level and j is the vertex of the $(k+1)$th level, then $d(i, j, D^{***}) = k.$ From here we finally have $diam(D^{***}) = k$. Since all possible cases have been considered, the theorem is proved.

Collorary 3.6. Proving the Seymour's conjecture for orgraphs of diameter 3 is equivalent to proving the Seymour's conjecture for all the orgraphs.

Proof. Assume the opposite, i.e., there is a counterexample to the conjecture, but there are no counterexamples of diameter 3. From Theorem 3.5 is follows that the existence of at least one counterexample implies the existence of a counterexample with diameter 3. This is a contradiction, so the initial assumption of the existence of at least one counterexample is false.

A discussion of the above-mentioned problems was submitted by me and my colleagues to a scientific journal and was published recently [10].

Theorem 3.7. If for the orgraph D with diameter 3 and with n vertices the inequality $\sum_{v \in V(D)} d^{+++}(v) \leq n$ holds, then D has a Seymour vertex.

Proof. Let $D = (V; A)$ have a diameter 3 and n vertices. We make the following substitutions:

$$\sum_{v \in V(D)} d^{-}(v) = k^{-}, \sum_{v \in V(D)} d^{+}(v) = k^{+}, \sum_{v \in V(D)} d^{++}(v) = k^{++}, \sum_{v \in V(D)} d^{+++}(v) = k^{+++}$$

Assume the opposite, i.e., D does not have a Seymour vertex. In this case, for each vertex $v \in V(D)$ the inequality $d^{+}(v) \geq d^{++}(v) + 1$ holds. Sum this inequality for all vertices of the orgraph D and obtain

$$k^{+} \geq k^{++} + n \iff k^{+} - k^{++} \geq n \tag{3.6}$$

In addition, for each $v \in V(D)$ the inequality $d^{-}(v) \leq d^{++}(v) + d^{+++}(v)$ holds, sice $N^{-}(v) \subseteq N^{++}(v) + N^{+++}(v)$. Because $k^{-} = k^{+}$ [1, c. 72], we have

$$k^{+++} \geq k^{-} - k^{++} = k^{+} - k^{++} \tag{3.7}$$

With using inequalities (3.6) and (3.7) we get $k^{+++} \geq n$, which contradicts the condition, except in the case $k^{+++} = n$. Consider it separately. It is clear that for any vertex $v \in V(D)$ $\quad n = 1 + d^{+}(v) + d^{++}(v) + d^{+++}(v)$, because every other vertex belongs belongs either to $N^{+}(v)$, or to $N^{++}(v)$, or to $N^{+++}(v)$. Sum this inequality for all vertices of graph D and obtain

$$n^2 = n + k^{+} + k^{++} + k^{+++} \iff n^2 - n = k^{+} + (k^{++} + n) \tag{3.8}$$

Substitute (3.6) into (3.8): $n^2 - n = k^{+} + (k^{++} + n) \leq k^{+} + + k^{+} = 2k^{+} \implies k^{+} \geq \frac{n(n-1)}{2}$. According to the substitution, k^{+} is the number of all arcs in D, and it cannot exceed $\frac{n(n-1)}{2}$, since this is the maximum possible number of arcs in a digraph with n vertices. If $k^{+} = \frac{n(n-1)}{2}$, then D is a tournament graph in which the Seymour vertex exists [5]. Thus we also obtain a contradiction that proves the presence of the Seymour vertex in the orgraph D. Theorem 3.7 is proved.

4. APPLYING PENALTY FUNCTION FOR ANALYSIS OF POSSIBLE COUNTEREXAMPLES

A counterexample to Seymour's conjecture (if it exists) with the fewest vertices among all other counterexamples is a strongly connected digraph [8].

Let $D = (V(D), A(D))$, and let $D^* = (V(D), A(D^*))$, where D^* is a subgraph of the digraph D with the same set of vertices ($A(D^*) \subseteq A(D)$).

Theorem 4.1. If $H(D^*) > 2(|A(D)| - |A(D^*)|)$, then D has the Seymour vertex.

Proof. Firstly, we shall prove the following lemma.

Lemma 4.2. Penalty of a digraph cannot be reduced by more than 2 after adding an arc to it.

Proof. Denote by $H_0(i)$ the penalty of the vertex before adding the arc, and by $H_1(i)$ the penalty after adding the arc. When we add an arc (i, j) to any graph, the vertex j increases $d^+(i)$ by 1. The vertex j could belong to $N^{++}(i)$ before this addition, and then become an element of $N^+(i)$. From this we conclude that $H_1(i) \geq (d^{++}(i) - 1) - (d^+(i) + 1) = (d^{++}(i) - d^+(i)) - 2 = H_0(i) - 2$, i.e., after adding an arc, the penalty of vertex i can be reduced by no more than 2. On the other hand, the penalty of other vertices has not decreased, because the penalty function depends on the change of the set N^+. Lemma 4.2 is proved.

From Lemma 4.2 when we add $|A(D)| - |A(D^*)|$ arcs to the digraph D^* its penalty can be reduced by no more than $2|A(D)| - |A(D^*)|$, i.e., $H(D) \geq H(D^*) - 2(|A(D)| - |A(D^*)|) > 0$. Theorem 4.1 is proved.

From Theorem 4.1 is follows that if for there exists at least one subgraph D^* of the digraph D with the same vertices, for which the condition $H(D^*) > 2(|A(D)| - |A(D^*)|)$ holds, D is not a counterexample to Seymour's conjecture with the minimum number of vertices.

Let $D = (V, A, S_1, S_2, \dots, S_k)$ be a directed graph consisting of nonempty subgraphs $S_1, S_2, \dots, S_k$ ($k \geq 3$). From each vertex of the subgraph S_i there are arcs going to all vertices of S_{i+1} ($1 \leq i \leq k$, we consider $S_{k+1} = S_1$). Such arcs will be called consecutive. The number of other arcs between subgraphs S_i does not exceed $\frac{k-1}{2}$. Such arcs will be called inconsistent.

Theorem 4.3. The graph $D = (V, A, S_1, S_2, \dots, S_k)$ is not a counterexample to Seymour's conjecture with the minimum number of vertices.

Proof. Assume the opposite, that the graph D is a counterexample with the minimum number of vertices. Let $V(S_i) = n_i$ (assume that $n_{k+1} = n_1, n_{k+2} = n_2$). Since in each of the subgraphs S_i there are fewer vertices than in graph D and D is a counterexample with the minimum number of vertices by assumption, then each of subgraphs S_i has a Seymour vertex. Denote the Seymour vertex for S_i by v_i. From the graph D we remove all inconsistent arcs, and we denote the new graph by D^*. In D^* we estimate the sum or penalties for $v_i \in V(D)$, $1 \le i \le k$: $d^+(v_i) = d^+(v_i, S_i) + n_{i+1}$, $d^{++}(v_i) = d^{++}(v_i, S_i) + n_{i+2}$. Because v_i is the Seymour vertex in the subgraph S_i, then $d^{++}(v_i, S_i) \ge d^+(v_i, S_i) \Rightarrow d^{++}(v_i) - d^+(v_i) \ge n_{i+2} - n_{i+1}$.

From the definition of the penalty function and the previous inequality we obtain $H(v_i) \ge n_{i+2} - n_{i+1} + 1,\ 1 \le i \le k$. Let $a_i = n_{i+2} - n_{i+1}$, $1 \le i \le k$, then $\sum_{i=1}^{k} a_i = 0$.

Now the previous inequality can be written as follows

$$H(v_i) \ge a_i + 1,\ 1 \le i \le k \tag{4.1}$$

After adding the sum of k inequalities (4.1) for different i, we have

$$\sum_{i=1}^{k} H(v_i) \ge \sum_{i=1}^{k} a_i + k = k$$

The penalty of a graph is not less than the penalty of its vertices:

$$H(D^*) \ge \sum_{i=1}^{k} H(\mathrm{v}_i) = k$$

In order to return to graph D from D^*, it is needed to add no more than $\frac{k-1}{2}$ arcs. When each arc is added, the penalty of the graph can be reduced by no more than 2 (Lemma 4.2). Therefore, $H(D) \ge H(D^*) - 2\frac{k-1}{2} \ge 1$. The penalty of the graph is positive, so the graph D is not a counterexample. We have received a contradiction that completes the proof.

5. VERTEX-WEIGHTED SEYMOUR'S CONJECTURE

Seymour's Second Neighborhood Conjecture can be considered on vertex-weighted digraphs. Such graphs have a weight function η, which assigns a positive real number to each vertex. This can be extended to finding the weight function on the set of vertices S, where

$$\eta(S) = \sum_{v \in S} \eta(v)$$

The vertex v for which $\eta\big(N^{++}(v)\big) \geq \eta\big(N^{+}(v)\big)$ is called the Seymour vertex.

Conjecture 5.1 (vertex-weighted Seymour's conjecture). Each vertex-weighted digraph without cycles of length two has a Seymour vertex.

Theorem 5.2. Seymour's conjecture is equivalent to Conjecture 5.1.

Proof. Assume that the vertex-weighted conjecture holds. Then there is a digraph D with weighted vertices, in which there is a Seymour vertex for which $\eta\big(N^{++}(v)\big) \geq$ $\geq \eta\big(N^{+}(v)\big)$ holds. If the weight of each vertex is equal to 1, i.e., $\eta(v) = 1$, then the previous inequality will be followed by $d^{++}(v) \geq d^{+}(v)$, so v is the Seymour vertex of Second Neighborhood Conjecture. Thus, from the validity of the vertex-weighted conjecture follows the validity of ordinary Seymour's conjecture.

Now assume that Seymour's conjecture hold, and we will prove that the vertex-weighted conjecture holds. Let the vertex-weighted digraph D have a weight function η and n vertices, $V(D) = \{v_1, v_2, \dots, v_n\}$.

Firstly, we consider the case when η takes natural values, i.e., $\eta(v) \in \mathbb{N}$. Construct an unweighted digraph D^*, which consists of n subgraphs $S_1, S_2, \dots, S_n$, and the subgraph S_i consists of $\eta(v_i)$ isolated vertices $\big(|V(S_i)| = \eta(v_i)\big)$. If $(v_i, v_j) \in A(D)$, then from each vertex of the subgraph S_i there is an arc to each vertex of the subgraph S_j. If $(v_i, v_j) \notin A(D)$, then in the digraph D^* there is no arc that begins at the vertex if the subgraph S_i and ends at the vertex of the subgraph S_j.

Since the Seymour's conjecture holds, the digraph D^* has the Seymour vertex v^*. Let $v^* \in V(S_i)$, then $d^{++}(v^*) \geq d^+(v^*)$. After returning to the graph D, we obtain that $\eta\big(N^{++}(v_i)\big) \geq \eta\big(N^+(v_i)\big)$, so the graph D has the Seymour vertex v_i, therefore, the vertex-weighted Seymour's conjecture holds.

Now we consider the case when η takes positive rational values, i.e., $\eta(v) \in \mathbb{Q}_+$. Let $\eta(v_i) = \frac{m_i}{\mathrm{n}_i}$, and let $p = \mathrm{LCM}(\mathrm{n}_1, \mathrm{n}_2, \dots, \mathrm{n}_k)$. We construct a new weight function $\eta^*(v_i) =$ $= p\eta(v_i)$. It is clear that $\eta^*(v_i) \in \mathbb{N}$, so by the previous case in the graph D with the weight function η^* there is a vertex v^*, for which

$$\eta^*\big(N^{++}(v^*)\big) \geq \eta^*\big(N^+(v^*)\big) \Leftrightarrow \frac{\eta^*\big(N^{++}(v^*)\big)}{p} \geq \frac{\eta^*\big(N^+(v^*)\big)}{p} \Leftrightarrow \eta\big(N^{++}(v^*)\big) \geq \eta\big(N^+(v^*)\big)$$

Therefore, the initial graph D with function η has a Seymour vertex.

Lastly, we consider the case when η takes positive real values, i.e., $\eta(v) \in \mathbb{R}_+$. Assume the opposite, that the graph D with weight function η does not have a Seymour vertex. Then for an arbitrary vertex v we have $\eta\big(N^+(v)\big) > \eta\big(N^{++}(v)\big)$. Let

$$\varepsilon = \min_{v \in V(D)} \{\, \eta\big(N^+(v)\big) - \eta\big(N^{++}(v)\big) \,\} \tag{5.1}$$

Let $\varepsilon^* = \frac{\varepsilon}{n}$, $\eta(v_i) = a_i$. Since the set of rational number is everywhere dense set in the set of real number [9], then between two real numbers a_i and $a_i + \varepsilon^*$ there is always a rational number. Denote this number by q_i, then $a_i < q_i < a_i + \varepsilon^*$. We construct a new weight function $\eta^*(v_i) = q_i$.

From the previous case graph D with the weight function η^* has a Seymour vertex v for which

$$\eta^*\big(N^+(v)\big) \leq \eta^*\big(N^{++}(v)\big) \Leftrightarrow \eta^*\big(N^+(v)\big) - \eta^*\big(N^{++}(v)\big) \leq 0$$

$$\eta\big(N^+(v)\big) - \eta\big(N^{++}(v)\big) \leq \eta\big(N^+(v)\big) - \eta\big(N^{++}(v)\big) - \Big(\eta^*\big(N^+(v)\big) - \eta^*\big(N^{++}(v)\big)\Big) =$$

$$= \Big(\eta\big(N^+(v)\big) - \eta^*\big(N^+(v)\big)\Big) + (\eta^*\big(N^{++}(v)\big) - \eta\big(N^{++}(v)\big)) < n\varepsilon^* = \varepsilon$$

Therefore, $\eta\big(N^+(v)\big) - \eta\big(N^{++}(v)\big) < \varepsilon$. We obtained a contradiction with (5.1), then the initial graph D has a Seymour vertex. Theorem 5.2 is proved.

6. CONCLUSIONS

In this project we considered the formulation of Seymour's Second Neighborhood Conjecture and characterized the level of research of the problem by the analysis of already proved statements for graphs with different properties.

The range of values of diameters and densities of possible counterexamples to the conjecture was analysed.

It is proved that if there is at least one counterexample to Seymour's conjecture, then there are counterexamples with an arbitrary diameter of at least 3, and there are counterexamples with both high (close to 1) and low (close to 0) density.

It is substantiated that the proof of the Seymour's conjecture in general follows from its proof only for the digraph of diameter 3.

We used the penalty function to prove that graphs with certain properties cannot be counterexample to Seymour's conjecture with minimum number of vertices.

It is proved that if one arc is added to the graph, then the total penalty of the graph can be reduced by no more than 2.

A generalized version of Seymour's conjecture is considered, in which each vertex has a certain positive weight.

The equivalence of Seymour's conjecture to vertex-weighted Seymour's conjecture is proved

Further research will focus on finding some other properties of possible counterexamples to the Seymour's conjecture, as well as proving the Seymour's conjecture for a digraph of diameter 3.

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This project is devoted to the study of Seymour's Second Neighborhood conjecture on digraphs. Mainly, the author tries to characterize the properties of possible counterexamples. Some results are obtained via considering the densities and diameters of such examples. One of the results shows that if the conjecture is true for digraphs of diameter 3, then the conjecture is verified. Moreover, the conjecture for weighted digraphs version is also verified. In a word, this is a solid research project. Hopefully, more results can be obtained in the near future.